

One-mode quantum Gaussian channels

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Abstract

A classification of one-mode Gaussian channels is given up to canonical unitary equivalence. A complementary to the quantum channel with additive classical Gaussian noise is described providing an example of one-mode Gaussian channel which is neither degradable nor anti-degradable.

1 The canonical form

For a mathematical framework of linear Bosonic system used in this note the reader is referred to [1]. Consider Bosonic system with one degree of freedom described by the canonical observables Q, P satisfying the Heisenberg canonical commutation relations (CCR)

$$[Q, P] = iI. \quad (1)$$

Let Z be the two-dimensional symplectic space, i. e. the linear space of vectors $z = [x, y]$ with the symplectic form

$$\Delta(z, z') = x'y - xy'. \quad (2)$$

A basis e, h in Z is symplectic if $\Delta(e, h) = 1$, i.e. if the area of the oriented parallelogram based on e, h is equal to 1. A linear transformation T in Z is symplectic if it maps a symplectic basis into symplectic basis, or equivalently

$$\Delta(Tz, Tz') = \Delta(z, z'); \quad z, z' \in Z.$$

Let $V(z) = \exp i(xQ + yP)$ be the unitary Weyl operators in a Hilbert space \mathcal{H} satisfying the CCR

$$V(z)V(z') = \exp\left[\frac{i}{2}\Delta(z, z')\right]V(z + z')$$

formally equivalent to (1). For any symplectic transformation T in Z there is a canonical unitary transformation U_T in \mathcal{H} such that

$$U_T^* V(z) U_T = V(Tz).$$

An arbitrary Gaussian channel Φ in $\mathfrak{B}(\mathcal{H})$ has the following action on the Weyl operators (we use the dual channel Φ^* in Heisenberg picture)

$$\Phi^*(V(z)) = V(Kz)f(z), \quad (3)$$

where K is a linear transformation in Z while $f(z)$ is a Gaussian characteristic function satisfying the condition that for arbitrary finite collection $\{z_r\} \subset Z$ the matrix with the elements

$$f(z_r - z_s) \exp\left(-\frac{i}{2}\Delta(z_r, z_s) + \frac{i}{2}\Delta(Kz_r, Kz_s)\right) \quad (4)$$

is positive definite. Considering the function $f(z)$, we can always eliminate the linear terms by a canonical transformation and assume that $f(z) = \exp\left[-\frac{1}{2}\alpha(z, z)\right]$, where α is a quadratic form. Then (4) is equivalent to positive definiteness of the matrix with the elements

$$\alpha(z_r, z_s) - \frac{i}{2}\Delta(z_r, z_s) + \frac{i}{2}\Delta(Kz_r, Kz_s). \quad (5)$$

Motivated by [2], [3], we are interested in the simplest form of one-mode Gaussian channel which can be obtained by applying suitable canonical unitary transformations to the input and the output of the channel:

$$\Phi^{*'}[V(z)] = U_{T_1}^* \Phi^* [U_{T_2}^* V(z) U_{T_2}] U_{T_1}$$

i.e.

$$\Phi^{*'}[V(z)] = V(T_1 K T_2 z) f(T_2 z).$$

Theorem. Let e, h be a symplectic basis; depending on the value

$$A) \quad \Delta(Ke, Kh) = 0; \quad B) \quad \Delta(Ke, Kh) = 1;$$

$$C) \quad \Delta(Ke, Kh) = k^2 > 0, k \neq 1; \quad D) \quad \Delta(Ke, Kh) = -k^2 < 0$$

there are symplectic transformations T_1, T_2 , such that $\Phi^{*'}$ has the form (3) with

$$\begin{aligned}
A_1) \quad K[x, y] &= [0, 0]; \\
f(z) &= \exp \left[-\frac{1}{2} \left(N_0 + \frac{1}{2} \right) (x^2 + y^2) \right]; \quad N_0 \geq 0; \\
A_2) \quad K[x, y] &= [x, 0]; \\
f(z) &= \exp \left[-\frac{1}{2} \left(N_0 + \frac{1}{2} \right) (x^2 + y^2) \right]; \\
B_1) \quad K[x, y] &= [x, y]; \\
f(z) &= \exp \left[-\frac{1}{4} x^2 \right]; \\
B_2) \quad K[x, y] &= [x, y]; \\
f(z) &= \exp \left[-\frac{1}{2} N_c (x^2 + y^2) \right]; \quad N_c \geq 0; \\
C) \quad K[x, y] &= [kx, ky]; \quad k > 0, k \neq 1; \\
f(z) &= \exp \left[-\frac{|k^2 - 1|}{2} \left(N_0 + \frac{1}{2} \right) (x^2 + y^2) \right]; \\
D) \quad K[x, y] &= [kx, -ky]; \quad k > 0; \\
f(z) &= \exp \left[-\frac{(k^2 + 1)}{2} \left(N_0 + \frac{1}{2} \right) (x^2 + y^2) \right].
\end{aligned}$$

Proof.

A) $\Delta(Ke, Kh) = 0$. In this case $\Delta(Kz, Kz') \equiv 0$ and either $K = 0$ or K has rank one. Then positive definiteness of (5) is just the condition on the correlation function of a quantum Gaussian state. As follows from Williamson theorem, see e. g. [1], there is a symplectic transformation T_2 such that

$$\alpha(T_2 z, T_2 z) = \left(N_0 + \frac{1}{2} \right) (x^2 + y^2).$$

In the case $K = 0$ we just have A_1).

Let K have rank one, then KT_2 has rank one and there is a vector e' such that $KT_2[x, y] = e'$. Then there is a symplectic transformation T_1 such that $T_1 e' = [1, 0]$ and hence $T_1 KT_2[x, y] = [k_1 x + k_2 y, 0]$. By making a rotation T_2' which leaves α unchanged, we can transform this vector to $[k_1' x, 0]$ with $k_1' \neq 0$, and then by a symplectic scaling (squeezing) T_1' we can transform to the case A_2).

B,C) $\Delta(Ke, Kh) = k^2 > 0$. Then $T = k^{-1}K$ is symplectic transformation and $\Delta(Kz, Kz') = k^2 \Delta(z, z')$. Let $T_1 = (TT_2)^{-1}$ where T_2 will be chosen later, so that $T_1 KT_2 = kI$. If $k = 1$ then positive definiteness of (5) is just positive definiteness of α . The transformation T_2 is chosen as follows:

In the case B) $k = 1$ and positive definiteness of (5) is just positive definiteness of α . Then we have the subcases:

B₂) If α is nondegenerate then by Williamson theorem there is a symplectic transformation T_2 such that

$$\alpha(T_2 z, T_2 z) = N_c (x^2 + y^2),$$

where $N_c > 0$. Also if $\alpha = 0$ we have a similar formula with $N_c = 0$.

B₁) On the other hand, if α is degenerate of rank one, that is $\alpha(z, z) = (k_1 x + k_2 y)^2$ for some k_1, k_2 simultaneously not equal to zero, then for arbitrary $N_c > 0$ there is a symplectic transformation T_2 such that

$$\alpha(T_2 z, T_2 z) = N_c y^2$$

in particular, we can take $N_c = \frac{1}{2}$.

In the case C) $k \neq 1$, and positive definiteness of (5) implies that $\alpha/|k^2 - 1|$ is correlation function of a quantum Gaussian state, hence .

$$\alpha(T_2 z, T_2 z) = |k^2 - 1| \left(N_0 + \frac{1}{2} \right) (x^2 + y^2)$$

with $N_0 \geq 0$.

D) $\Delta(Ke, Kh) = -k^2 < 0$. Then $T = k^{-1}K$ is antisymplectic transformation ($\Delta(Tz, Tz') = -\Delta(z, z')$) and $\Delta(Kz, Kz') = -k^2\Delta(z, z')$. Similarly to the case B,C) we obtain

$$\alpha(T_2 z, T_2 z) = |k^2 + 1| \left(N_0 + \frac{1}{2} \right) (x^2 + y^2)$$

with $N_0 \geq 0$. Letting $T_1 = \Lambda(TT_2)^{-1}$, where $\Lambda[x, y] = [x, -y]$, we obtain the first equation in D). Note that T_1 is symplectic since both T and Λ are antisymplectic. \square

2 Description in terms of open Bosonic system

As explained in [1], a Gaussian channel Φ can be dilated to a linear dynamics (i. e. symplectic transformation) of open Bosonic system described by Q, P and ancillary canonical variables q, p, \dots in a Gaussian state ρ_0 ; moreover this linear dynamics provides also a description of the channel Φ_E mapping initial state of the system Q, P into the final state of the ancilla (environment) q, p, \dots ; in the case where the state ρ_0 is pure, Φ_E is just the complementary channel $\tilde{\Phi}$ in the sense of [5], [4], [6], which is determined by Φ up to unitary equivalence. In the case of arbitrary ρ_0 , following [7], we will call the channel Φ_E weak complementary.

Let us give this description in each of the cases of the previous Section.

A₁) This is completely depolarizing channel

$$\begin{aligned} Q &\rightarrow q \\ P &\rightarrow p \end{aligned}$$

where q, p are in the quantum thermal state ρ_0 with mean number of quanta N_0 . Its weak complementary is the ideal channel Id.

A₂) The linear transformation of the mode canonical variables is

$$\begin{aligned} Q &\rightarrow Q + q \\ P &\rightarrow p, \end{aligned} \tag{6}$$

where q, p are in the quantum thermal state ρ_0 with mean number of quanta N_0 . The signal in this channel is “classical” in that it has commuting components, so its quantum capacity must be equal to zero. In fact, it can be shown [7] that this channel is anti-degradable¹, hence the conclusion follows. Weak complementary of this channel is described by the transformation

$$\begin{aligned} q &\rightarrow Q \\ p &\rightarrow P - p, \end{aligned} \tag{7}$$

where p can be regarded as a classical real Gaussian variable with variance $N_0 + \frac{1}{2}$.

B₁) The equation of the channel has the form (7) where p has variance $\frac{1}{2}$, so that the mode q, p is in pure (vacuum) state, and the complementary channel is given by (6), so that by the previous remark, the channel is degradable. Its quantum capacity is infinite as can be seen from the lower bound given by the Gaussian coherent information resulting from the Gaussian input states with $\sigma_P^2 = DP \rightarrow \infty$. In more detail, denoting the channel given by the equation (7) by Φ and its complementary – by $\tilde{\Phi}$ we have for the coherent information $J(\rho, \Phi) = H(\Phi[\rho]) - H(\tilde{\Phi}[\rho])$. Now let ρ be the Gaussian state with $\sigma_Q^2 = DQ, \sigma_P^2 = DP$ and zero means and covariance, then $\Phi[\rho]$ has the correlation matrix

$$\begin{bmatrix} \sigma_Q^2 & 0 \\ 0 & \sigma_P^2 + \frac{1}{2} \end{bmatrix},$$

while $\tilde{\Phi}[\rho]$ has the correlation matrix

$$\begin{bmatrix} \sigma_Q^2 + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

By a formula from [1],

$$\begin{aligned} H(\Phi[\rho]) &= g\left(\sqrt{\sigma_Q^2\left(\sigma_P^2 + \frac{1}{2}\right)} - \frac{1}{2}\right), \\ H(\tilde{\Phi}[\rho]) &= g\left(\sqrt{\left(\sigma_Q^2 + \frac{1}{2}\right)\frac{1}{2}} - \frac{1}{2}\right), \end{aligned}$$

where $g(x) = (x + 1)\log(x + 1) - x\log x$. In particular, taking states with $\sigma_Q^2\sigma_P^2 \rightarrow \infty, \sigma_Q^2 \rightarrow 0$, we have $J(\rho, \Phi) \rightarrow \infty$.

¹Channel Φ is *degradable* [5] if $\tilde{\Phi} = T \circ \Phi$ for some channel T . It is called *anti-degradable* [2] if $\Phi = T' \circ \tilde{\Phi}$ for some channel T' .

B₂) Channel with additive complex Gaussian noise of intensity N_c

$$\begin{aligned} Q &\rightarrow Q + \xi \\ P &\rightarrow P + \eta \end{aligned}$$

to be described in detail in Sec. 3.

C) Attenuation/amplification channel with coefficient k and quantum noise with mean number of quanta N_0 . In the attenuation case ($k < 1$) the equation of the channel is

$$\begin{aligned} Q &\rightarrow kQ + \sqrt{1-k^2}q \\ P &\rightarrow kP + \sqrt{1-k^2}p, \end{aligned}$$

where q, p are in the quantum thermal state ρ_0 with mean number of quanta N_0 . The weak complementary is given by the equations

$$\begin{aligned} q &\rightarrow \sqrt{1-k^2}Q - kq \\ p &\rightarrow \sqrt{1-k^2}P - kp, \end{aligned}$$

and is again an attenuation channel (with coefficient $k' = \sqrt{1-k^2}$) see [4], [2].

In the amplification case ($k > 1$) we have

$$\begin{aligned} Q &\rightarrow kQ + \sqrt{k^2-1}q \\ P &\rightarrow kP - \sqrt{k^2-1}p, \end{aligned}$$

with the weak complementary

$$\begin{aligned} q &\rightarrow \sqrt{k^2-1}Q + kq \\ p &\rightarrow -\sqrt{k^2-1}P + kp, \end{aligned}$$

see the case D).

Alternatively, by introducing $N_c = |k^2-1|N_0 \geq 0$, the case C) is the same as attenuation/amplification channel with vacuum ancilla state and additive classical Gaussian noise of intensity N_c considered in [1], where one-shot Gaussian coherent information for this channel was computed. From [2] it follows that in the case $N_0 = 0$ the channel C) is degradable, hence the coherent information is subadditive in this case by [5] and the quantum capacity is equal to the maximum of one-shot coherent information. Moreover, it was observed in [8] that the one-shot coherent information is concave for degradable channels, hence it is maximized by a Gaussian state. To sum up, the quantum capacity of the attenuation/amplification channel with $N_0 = 0$ and coefficient k is equal to the maximized one-shot Gaussian coherent information $\max\{0, \log \frac{k^2}{|k^2-1|}\}$, an expression following from [1]. The case C) with $N_0 > 0$ as well as B₂) remain open questions.

D) The channel equation is

$$\begin{aligned} Q &\rightarrow kQ + \sqrt{k^2 + 1}q \\ P &\rightarrow -kP + \sqrt{k^2 + 1}p, \end{aligned}$$

which is the same as the weak complementary to amplification channel with coefficient $k' = \sqrt{k^2 + 1}$ and quantum noise with mean number of quanta N_0 , see [4] , [2]. It was shown in [2] that this channel is anti-degradable, hence has zero quantum capacity.

3 Quantum signal plus classical Gaussian noise

Here we give explicit construction of complementary channel in case B₂) based on corrected and improved presentation of Appendix B in [1].

Introducing the one mode annihilation operator $a = \frac{1}{\sqrt{2}}(Q + iP)$ consider the transformation

$$a' = a + \zeta, \quad (8)$$

where $\zeta = \frac{1}{\sqrt{2}}(\xi + i\eta)$ is a complex Gaussian random variable with zero mean and variance N_c . This means that in the plane of the complex variable $z = \frac{1}{\sqrt{2}}(x + iy)$ it has the probability distribution $\mu_{N_c}(d^2z)$ with the density

$$p(z) = (\pi N_c)^{-1} \exp(-|z|^2/N_c)$$

This transformation generates the channel

$$\Phi^*[f(a, a^\dagger)] = \int f(a + z, (a + z)^\dagger) p(z) d^2z,$$

in the Heisenberg picture, while the channel in the Schrödinger picture can be described by the formula

$$\Phi[\rho] = \int D(z) \rho D(z)^* p(z) d^2z, \quad (9)$$

where $D(z) = \exp(za^\dagger - \bar{z}a) = \exp i(yQ - xP) = V(-Jz)$ is the displacement operator. Here $J[x, y] = [-y, x]$ is the operator of complex structure. The channel describes the quantum mode Q, P in the additive classical Gaussian environment ζ i. e. the case B₂) of Sec. 1.

Here we construct the ancilla representation of the channel as unitary evolution of the mode Q, P interacting with quantum environment (ancilla), and find the complementary channel for Φ . We also argue that for certain values of N_c both channels have positive quantum capacities and hence are neither degradable nor anti-degradable, cf. [2].

First we need to extend the classical environment to a quantum system in a pure state. Consider the ancilla Hilbert space $\mathcal{H}_E = L^2(\mu_{N_c})$ with the vector

$|\Psi_0\rangle$ given by the function identically equal to 1. The tensor product $\mathcal{H} \otimes \mathcal{H}_E$ can be realized as the space $L^2_{\mathcal{H}}(\mu_{N_c})$ of μ_{N_c} -square integrable functions $\psi(z)$ with values in \mathcal{H} . Define the unitary operator U in $\mathcal{H} \otimes \mathcal{H}_E$ by

$$(U\psi)(z) = D(z)\psi(z). \quad (10)$$

Then

$$\Phi[\rho] = \text{Tr}_{\mathcal{H}_E} U (\rho \otimes |\Psi_0\rangle\langle\Psi_0|) U^*,$$

while the complementary channel is

$$\Phi_E[\rho] = \tilde{\Phi}[\rho] = \text{Tr}_{\mathcal{H}} U (\rho \otimes |\Psi_0\rangle\langle\Psi_0|) U^*.$$

This means that $\tilde{\Phi}[\rho]$ is an integral operator in $L^2(\mu_{N_c})$ with the kernel

$$K(z, z') = \text{Tr} D(z) \rho D(z')^*.$$

In case of Gaussian state ρ_N with the characteristic function

$$\text{Tr} \rho_N D(z) = \exp(-(N+1/2)|z|^2) \quad (11)$$

we have

$$K(z, z') = \exp(i\Im \bar{z}' z - (N+1/2)|z - z'|^2).$$

Let us define the canonical observables in $L^2(\mu_{N_c})$. For this consider the unitary Weyl operators $V(z_1, z_2)$ in $L^2(\mu_{N_c})$ by

$$\begin{aligned} (V(z_1, z_2)\psi)(z) &= \psi(z + z_2) \\ &\times \exp \left[i2\Re \bar{z}_1(z + \frac{z_2}{2}) - \frac{1}{N_c} \Re \bar{z}_2(z + \frac{z_2}{2}) \right]. \end{aligned} \quad (12)$$

The operators $V(z_1, z_2)$ satisfy the Weyl-Segal CCR

$$V(z_1, z_2)V(z'_1, z'_2) = \exp[\frac{i}{2}\Delta((z_1, z_2), (z'_1, z'_2))]V(z_1 + z'_1, z_2 + z'_2), \quad (13)$$

with two degrees of freedom with respect to the symplectic form

$$\Delta((z_1, z_2), (z'_1, z'_2)) = 2\Re(\bar{z}'_1 z_2 - \bar{z}_1 z'_2) = x'_1 x_2 - x_1 x'_2 + y'_1 y_2 - y_1 y'_2,$$

where $z_j = \frac{1}{\sqrt{2}}(x_j + iy_j)$; $j = 1, 2$. By the general theory of CCR (see e.g. [1]) one has

$$V(z_1, z_2) = \exp i(x_1 q_1 + y_1 q_2 + x_2 p_1 + y_2 p_2),$$

where the canonical observables q_j, p_j ; $j = 1, 2$, satisfy the Heisenberg CCR

$$[q_j, p_k] = i\delta_{jk}I, \quad [q_j, q_k] = 0, \quad [p_j, p_k] = 0,$$

and commute with Q, P . By differentiating (12) with respect to x_1, x_2 at zero point, we obtain that

$$(q_1\psi)(z) = x\psi(z); \quad (q_2\psi)(z) = y\psi(z). \quad (14)$$

The state of ancilla is pure and given by the function in $L^2(\mu_{N_c})$ identically equal to one. It has the characteristic function

$$\begin{aligned} & \int (V(z_1, z_2)1)(z)p(z)d^2z \\ &= (\pi N_c)^{-1} \int \exp \left[i2\Re \bar{z}_1(z + \frac{z_2}{2}) - \frac{1}{N_c}\Re \bar{z}_2(z + \frac{z_2}{2}) - \frac{|z|^2}{N_c} \right] d^2z \\ &= \exp \left[-N_c|z_1|^2 - \frac{1}{4N_c}|z_2|^2 \right], \end{aligned}$$

which means that it is pure Gaussian with $q_j, p_j; j = 1, 2$ having zero means, zero covariances and the variances

$$\mathbb{D}q_1 = \mathbb{D}q_2 = N_c; \quad \mathbb{D}p_1 = \mathbb{D}p_2 = \frac{1}{4N_c}.$$

The Heisenberg dynamics of the Weyl operators of system+ancilla is given by the equations

$$D(z_0) \rightarrow D(z)^* D(z_0) D(z) = \exp 2i\Im(\bar{z}z_0) D(z_0), \quad (15)$$

$$V(z_1, z_2) \rightarrow D(z)^* V(z_1, z_2) D(z) = \exp i\Im(\bar{z}z_2) D(z_2) V(z_1, z_2), \quad (16)$$

where the first is the CCR for the displacement operators, while the second follows from (10), (12). By differentiating these relations with respect to $x_j, y_j; j = 0, 1, 2$, and taking into account (14) we obtain the Heisenberg equations for the canonical observables

$$\begin{aligned} Q &\rightarrow Q + q_1, \\ P &\rightarrow P + q_2, \\ q_1 &\rightarrow q_1, \\ p_1 &\rightarrow p_1 - P - q_2/2, \\ q_2 &\rightarrow q_2, \\ p_2 &\rightarrow p_2 + Q + q_1/2. \end{aligned}$$

Together with the ancilla state described above, the first two equations are the same as the equation (8) determining the channel Φ , while the last four equations give the action of the complementary channel $\tilde{\Phi}$ in terms of the canonical observables.

The characteristic function of the output ancilla state $\tilde{\Phi}[\rho]$ is

$$\text{Tr} \tilde{\Phi}[\rho] V(z_1, z_2) = \text{Tr} \rho D(z_2) \exp \left[- \left(N_c |z_1|^2 + N_c \Im \bar{z}_1 z_2 + \frac{N_c^2 + 1}{4N_c} |z_2|^2 \right) \right]. \quad (17)$$

Proof of (17). By using (16) we have

$$\begin{aligned} \text{Tr} \tilde{\Phi}[\rho] V(z_1, z_2) &= \int \text{Tr} D(z)^* V(z_1, z_2) D(z) \rho p(z) d^2z \\ &= \int \exp i\Im(\bar{z}z_2) \text{Tr} \rho D(z_2) (V(z_1, z_2)1)(z) p(z) d^2z \end{aligned}$$

which is equal to $\text{Tr} \rho D(z_2)$ multiplied by the integral

$$\int \exp \left[i\Im(\bar{z}z_2) + i2\Re\bar{z}_1(z + \frac{z_2}{2}) - \frac{1}{N_c}\Re\bar{z}_2(z + \frac{z_2}{2}) - \frac{|z|^2}{N_c} \right] \frac{d^2z}{\pi N_c}.$$

By introducing the change of variables $z + \frac{z_2}{2} = w$, we obtain that this is equal to

$$\begin{aligned} & \exp \left[-\frac{|z_2|^2}{4N_c} \right] \int \exp \left[i\Im(\bar{w}z_2) + i2\Re\bar{z}_1w - \frac{|w|^2}{N_c} \right] \frac{d^2w}{\pi N_c} \\ = & \exp \left[-\frac{|z_2|^2}{4N_c} \right] \int \exp \left[i2\Re\bar{w} \left(z_1 - i\frac{z_2}{2} \right) - \frac{|w|^2}{N_c} \right] \frac{d^2w}{\pi N_c} \end{aligned}$$

where the last integral is just the characteristic function of the complex Gaussian distribution, equal to

$$\exp \left[-N_c \left| z_1 - i\frac{z_2}{2} \right|^2 \right] = \exp \left[-N_c \left(|z_1|^2 + \Im\bar{z}_1z_2 + \frac{1}{4}|z_2|^2 \right) \right]$$

whence (17) follows. \square

In the case of Gaussian $\rho = \rho_N$ given by (11), we obtain

$$\text{Tr} \tilde{\Phi}[\rho_N] V(z_1, z_2) = \exp \left[- \left(N_c |z_1|^2 + N_c \Im\bar{z}_1z_2 + \frac{D^2}{4N_c} |z_2|^2 \right) \right],$$

where $D = \sqrt{(N_c + 1)^2 + 4N_c N}$, so that $\frac{D^2}{4N_c} = \frac{1}{4}(N_c + \frac{1}{N_c}) + (N + \frac{1}{2})$, which is Gaussian characteristic function with the correlation matrix

$$\alpha'_E = \begin{bmatrix} N_c & 0 & 0 & N_c/2 \\ 0 & N_c & -N_c/2 & 0 \\ 0 & -N_c/2 & \frac{D^2}{4N_c} & 0 \\ N_c/2 & 0 & 0 & \frac{D^2}{4N_c} \end{bmatrix}$$

Thus, with the commutation matrix of the ancilla

$$\Delta_E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

we obtain

$$\Delta_E^{-1} \alpha'_E = \begin{bmatrix} 0 & N_c/2 & -\frac{D^2}{4N_c} & 0 \\ -N_c/2 & 0 & 0 & -\frac{D^2}{4N_c} \\ N_c & 0 & 0 & N_c/2 \\ 0 & N_c & -N_c/2 & 0 \end{bmatrix}.$$

which has the eigenvalues $\lambda_\epsilon = \pm \frac{i}{2} (N_c \pm D)$; $\epsilon = 1, 2, 3, 4$.

Let the input state be Gaussian ρ_N given by (11), then the entropy of ρ_N is $H(\rho_N) = g(N)$. The output state has the entropy

$$H(\Phi[\rho_N]) = g(N + N_c), \quad (18)$$

while the exchange entropy is

$$\begin{aligned} H(\rho_N, \Phi) &= H(\tilde{\Phi}[\rho_N]) \\ &= \frac{1}{2} \sum_{\epsilon=1}^4 g\left(|\lambda_\epsilon| - \frac{1}{2}\right) \\ &= g\left(\frac{D + N_c - 1}{2}\right) + g\left(\frac{D - N_c - 1}{2}\right), \end{aligned} \quad (19)$$

see [1] for detail.

Introducing the coherent information $J(\rho, \Phi) = H(\Phi[\rho]) - H(\tilde{\Phi}[\rho])$, we have lower bounds for the quantum capacities $Q(\Phi)$, $Q(\tilde{\Phi})$:

$$\begin{aligned} \sup_{\rho} J(\rho, \Phi) &\leq Q(\Phi), \\ \sup_{\rho} [-J(\rho, \Phi)] &\leq Q(\tilde{\Phi}). \end{aligned}$$

Therefore if $J(\rho, \Phi)$ accepts the values of different signs, both capacities are positive, hence by using results from [2], the channel Φ is neither anti-degradable nor degradable. Consider the function

$$F(N) = J(\rho_N, \Phi) = g(N + N_c) - g\left(\frac{D + N_c - 1}{2}\right) - g\left(\frac{D - N_c - 1}{2}\right), \quad N \geq 0.$$

We have $F(0) = 0$; by using the asymptotic $g(x) \simeq \log ex$, $x \rightarrow \infty$, we obtain

$$\lim_{N \rightarrow \infty} F(N) = -\log eN_c,$$

hence $\lim_{N \rightarrow \infty} F(N) = 0$ if $N_c = e^{-1}$. Considering the graph of $F(N)$ for $N_c = 0.99e^{-1}$, one can see that $F(N)$ has a small negative peak for small positive N , while $\lim_{N \rightarrow \infty} F(N) = -\log 0.99 > 0$. Thus $F(N)$ and hence $J(\rho, \Phi)$ accepts the values of different signs, and the channel Φ is neither anti-degradable nor degradable.

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